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Partially smooth linear pretopological and topological operators for fuzzy sets

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Abstract

We introduce here pretopological operators for closure ($\mathcal{C}_{\alpha}^{\gamma}$) and interior ($\mathcal{I}_{\alpha}^{\gamma}$) based on partially smooth linear maps applied on the membership functions of any fuzzy set. Through a couple of theorems we also describe explicitly the correspondent topologies defined by the pretopological operators. The results from the current paper will be further extended in two directions - application in machine learning like clustering and classification, and generalization in the framework of intuitionistic fuzzy sets.

Keywords: Topology, Pretopology, Fuzzy Set

1 Introduction.

Three main notions will be discussed and unified in the presented paper - fuzzy sets, pretopological and topological spaces. We employ linear maps to introduce specific pretopological closure and interior operators and fuzzy set α-cuts for their topological counterparts. The results from this work will be further extended in a next research in two directions - application and generalization. We are going to state applications of the pretopological operators in machine learning like fuzzy clustering (cf. Yang [14]) and classification (cf. Amo et al. [2]). A generalization of the proposed pretopological and topological structure will be also further extended in the framework of the Intuitionistic Fuzzy Sets, where besides the degree of membership of a fuzzy set, there is also a degree of non-membership (cf. Atanassov [4], [3]).

The notion *fuzzy set* (FS) was introduced by Lotfi Zadeh in 1956 (cf. Zadeh [15]). A fuzzy set is an object whose elements memberships are not precisely defined. Fuzzy sets provide a better representation of reality than the classical mathematical binary representation of whether an element does or does not belong to a set. The membership in fuzzy sets is gradual, taking values in the range between "no" (0) and "yes" (1), that makes the theory invaluable to represent the limited level of precision in real world situations and preferences.

Formally, a fuzzy set in X (cf. Zadeh [15]), i.e. $A \in FS(X)$, is given by

$$A = \{\langle x, \mu_A(x)\rangle | x \in X\} \tag{1}$$

where

$$\mu_A : X \to [0, 1] \tag{2}$$

is the *membership function* of the fuzzy set A. The operations of inclusion, union, intersection, complement are extended from the ordinary to the fuzzy set theory. They are actually needed the notion of *fuzzy topological spaces* to be introduced (cf. Chang [7]). Since the introduction of fuzzy topological spaces there has been extensive research in the domain, e.g. Lowen [12], Chattopadhyay [8] and many others.

Since the introduction of fuzzy sets there have been many generalizations. Most of them consist of replacing the range $[0, 1]$ by more general algebraic structures satisfying the axioms for a lattice (cf. Birkhoff [6]) - they are called *L-fuzzy sets* (cf. Goguen [9]). Very popular extension of fuzzy sets is the Atanassov's generalization - *intuitionistic fuzzy* set (IFS) (cf. Atanassov [3], [4]). In addition to the membership function μ_A of $A \in IFS(X)$, there is another function ν_A, expressing a notion of non-membership degree with the same domain X and range $[0, 1]$, satisfying $0 \le \mu_A(x) + \nu_A(x) \le 1$ for every $x \in X$.

The second concept we are going to discuss in this paper is *topology* (Kuratowski [11], Arkhangelskii and Fedorchuk [1]). Many mathematicians and scientists actively employ concepts of topology to model and understand real-world structures and problems. A rich variety of results also has emerged in other areas of applied mathematics stemming from pure topological investigations. As topology originally grew up from geometry, it is often described as a rubber-sheet geometry - it means, literally, the study of position or location of points (elements) belonging to a given set called *topological space*. In traditional geometry, objects such as circles, polyhedra, triangles etc. are considered as rigid figures (bodies), with well-defined distances between points and angles between edges or faces. In comparison, in topology objects are considered as if they were made of rubber, capable of being deformed, which means that angles and distances are irrelevant. Objects can be bent, twisted, stretched, shrunk, or otherwise deformed from one to another, but we do not allow the objects to be ripped apart. Distances are not relevant in the concepts of topology but the notion of *proximity* is a very important concept, which is established by specifying a collection of subsets of the considered topological space called *open sets*. Open sets and their counterparts - closed sets in a topological space are often defined by interior and closure operators. What a topologist can do is identify and use the properties of objects that different shapes have in common. Often, in some situations, the properties that are significant are those that are preserved when we treat objects as deformable, as opposed when we treat them as rigid bodies. Such specific situations emerge in many areas of applied mathematics, physics, biology, geographic information systems and system theory.

Metric spaces and general topology are applicable in systems where geometry, differential and integral equations are essential in formulating models. But in systems described mainly by logic and algebra topological axioms are too restrictive. In 1967 Hammer [10] showed that by some extensions of the principles of general topology it is possible to model almost any formal system. He introduces the *isotonic spaces* which are later called *pretopogical spaces* (Arkhangelskii A. and Fedorchuk V. [1], Ch. 2.5). In 1981, inspired by the work of Hammer, Robert Badard [5] introduced the *fuzzy pretopological spaces* and showed many properties. We are going to state the axioms of fuzzy (pre)topological spaces and give concrete examples of such spaces, which can be applied in modeling real world problems. Our model will be actually generalization of the models stated in [13] by Sh. Wenzhong and L. Kimfung.

2 Pretopological and topological operators

Let us define first the preclosure and preinterior operators. We consider the universe X. X is a pretopological space in respect to the *preclosure* operator $\mathfrak{c}\colon \mathcal{X} \to \mathcal{X}$, where $\mathcal{X}$ can be $\mathcal{P}(X), FS(X)$ or $IFS(X)$ iff for any $A, B \in \mathcal{X}$ the following axioms are satisfied (cf [1], Ch. 2.5 and [5]).

1. $\mathfrak{c}(\emptyset) = \emptyset$
2. $A \subseteq \mathfrak{c}(A)$
3. $\mathfrak{c}(A \cup B) = \mathfrak{c}(A) \cup \mathfrak{c}(B)$

If in addition to the above stated axioms the operator $\mathfrak{c}$ is *idempotent*, that is $\mathfrak{c}(A) = \mathfrak{c}(\mathfrak{c}(A))$, then $\mathfrak{c}$ is called *closure* operator in $\mathcal{X}$.

Definition 1 (Fixed point) *Let $f : Z \longrightarrow Z$ be a function, where Z is an arbitrary set. Then $z_0 \in Z$ is called fixed point if $f(z_0) = z_0$.*

Remark 1 *The preclosure operator $\mathfrak{c}$ is idempotent iff for every $A \in \mathcal{X}$: $\mathfrak{c}(A)$ is a fixed point for $\mathfrak{c}$, i.e. $\mathfrak{c}(\mathfrak{c}(A)) = \mathfrak{c}(A)$.*

An example of (idempotent) closure operator is given in [3] and [4] (Ch. 1.6) in the case of IFSs, defined by

$$C(A) = \{\langle x, \max_{y \in X} \mu_A(y), \min_{y \in X} \nu_A(y)\rangle \mid x \in X\}.$$

Proposition 1 *The preclosure operator $\mathfrak{c}$ is non-decreasing in respect to the partial ordering $\subseteq$ in $\mathcal{X}$. That is, for all $A, B \in \mathcal{X}$,*

$$A \subseteq B \Rightarrow \mathfrak{c}(A) \subseteq \mathfrak{c}(B).$$

Proof Since $B = A \cup B$ and from the second axiom for preclosure $\mathfrak{c}(A \cup B) = \mathfrak{c}(A) \cup \mathfrak{c}(B)$, then

$$\mathfrak{c}(B) = \mathfrak{c}(A \cup B) = \mathfrak{c}(A) \cup \mathfrak{c}(B) \supseteq \mathfrak{c}(A).$$

Definition 2 (see [1], Ch. 2.5) *For the preclosure $\mathfrak{c}$ defined on $\mathcal{X}$ we say that a set $A \in \mathcal{X}$ is closed iff $\mathfrak{c}(A) = A$. That is, the closed sets are exactly the fixed points of $\mathfrak{c}$ and*

$$\tau^{\mathfrak{c}} = \{A \in \mathcal{X} \mid \mathfrak{c}(A) = A\},$$

is the topology generated by the preclosure operator $\mathfrak{c}$. If $\mathcal{X}$ is $\mathcal{P}(X), FS(X)$ and $IFS(X)$, then τ is called crisp topology, fuzzy topology and intuitionistic fuzzy topology, respectively.

There is a very important property for the opens sets, and namely,

Theorem 1 *For the preclosure operator $\mathfrak{c}$ defined on $\mathcal{X}$ and every family of closed sets $B_j \in \tau^{\mathfrak{c}}, j \in J$ their intersection is also a closed set. That is,*

$$(\bigcap_{j\in J} B_j) \in \tau^{\mathfrak{c}}.$$

Proof Let us take $B_j \in \tau^{\mathfrak{c}}, j \in J$ to be closed sets and we have to prove that $\mathfrak{c}(\bigcap_{j\in J} B_j) = (\bigcap_{j\in J} B_j)$. But since $\mathfrak{c}(\bigcap_{j\in J} A_j) \supseteq (\bigcap_{j\in J} B_j)$ follows by the second axiom for preinterior, then it is enough to show that $(\bigcap_{j\in J} B_j) \supseteq \mathfrak{c}(\bigcap_{j\in J} B_j)$. For any $j \in J, B_j = \mathfrak{c}(B_j) \supseteq \mathfrak{c}(\bigcap_{j\in J} B_j)$ from Proposition 2. We can take the intersection over all $j \in J$ in the left-hand side. And therefore $(\bigcap_{j\in J} B_j) \supseteq \mathfrak{c}(\bigcap_{j\in J} B_j)$, which proves the theorem.

For every preclosure operator on $\mathcal{X}$, and any $B \subseteq \mathcal{X}$ we define the *closure* of B,

$$Cl_{\mathfrak{c}}(B) = \bigcap\{B_1 \mid B_1 \supseteq B \ \&\ \mathfrak{c}(B_1) = B_1\}. \tag{3}$$

But Theorem 1 implies that $Cl_{\mathfrak{c}}(B)$ is closed set for every $B \subseteq \mathcal{X}$. And obviously $Cl_{\mathfrak{c}}(B)$ is the smallest closed set containing B.

Remark 2 *For every $B \subseteq \mathcal{X}$ and preclosure operator $\mathfrak{c}$ in $\mathcal{X}$,*

$$B = \mathfrak{c}^0(B) \subseteq \mathfrak{c}^1(B) \subseteq \mathfrak{c}^2(B) \subseteq \cdots \subseteq Cl_{\mathfrak{c}}(B).$$

and $Cl_{\mathfrak{c}}(B)$ is the smallest closed set containing $\mathfrak{c}^m(B)$, for all $m \in \mathbb{N}$.

Analogically to $\mathfrak{c}$, we define $\mathfrak{i}$, the *preinterior* and *interior* operators in $\mathcal{X}$. $\mathfrak{i}$ is preinterior operator if the following axioms are satisfied (see [1], Ch. 2.5),

1. $\mathfrak{i}(\mathcal{X}) = \mathcal{X}$
2. $\mathfrak{i}(A) \subseteq A$
3. $\mathfrak{i}(A \cap B) = \mathfrak{i}(A) \cap \mathfrak{i}(B)$

If in addition to the above stated axioms the operator $\mathfrak{i}$ is *idempotent*, that is $\mathfrak{i}(A) = \mathfrak{i}(\mathfrak{i}(A))$, then $\mathfrak{i}$ is called *interior* operator in $\mathcal{X}$ and $\mathcal{X}$ equipped with this operator has a topological structure. And again,

Remark 3 *The preinterior operator $\mathfrak{i}$ is idempotent if for every $A \in \mathcal{X}$: $\mathfrak{i}(A)$ is a fixed point for $\mathfrak{i}$.*

An example of (idempotent) interior operator is given in [3] and [4] (Ch. 1.6) in the case of IFSs, defined by

$$I(A) = \{\langle x, \min_{y\in X} \mu_A(y), \max_{y\in X} \nu_A(y)\rangle \mid x \in X\}.$$

Proposition 2 *The preinterior operator* $\mathfrak{i}$ *is non-decreasing in respect to the partial ordering* $\subseteq$ *in* $\mathcal{X}$*. That is, for all* $A, B \in \mathcal{X}$*,*

$$A \subseteq B \Rightarrow \mathfrak{i}(A) \subseteq \mathfrak{i}(B).$$

Proof The proof is almost straightforward counterpart of Proposition 1. Since $A = A \cap B$ and from the second axiom for preinterior $\mathfrak{i}(A \cap B) = \mathfrak{i}(A) \cap \mathfrak{i}(B)$, then

$$\mathfrak{i}(A) = \mathfrak{i}(A \cap B) = \mathfrak{i}(A) \cap \mathfrak{i}(B) \subseteq \mathfrak{i}(B).$$

Definition 3 (see [1], Ch. 2.5) *For the preinterior* $\mathfrak{i}$ *defined on* $\mathcal{X}$ *we say that a set* $A \in \mathcal{X}$ *is open iff* $\mathfrak{i}(A) = A$*. That is, the open sets are exactly the fixed points of* $\mathfrak{i}$ *and*

$$\tau_{\mathfrak{i}} = \{A \in \mathcal{X} \mid \mathfrak{i}(A) = A\},$$

is the topology generated by the preinterior operator $\mathfrak{i}$*. If* $\mathcal{X}$ *is* $\mathcal{P}(X)$ *or* $FS(X)$*, then* τ *is called crisp topology or fuzzy topology, respectively.*

There is a very important property for the opens sets, and namely,

Theorem 2 *For the preinterior* $\mathfrak{i}$ *defined on* $\mathcal{X}$ *and every family of open sets* $A_j \in \tau_{\mathfrak{i}}, j \in J$ *their union is also an open set. That is,*

$$(\bigcup_{j\in J} A_j) \in \tau_{\mathfrak{i}}.$$

Proof Let us take $A_j \in \tau_{\mathfrak{i}}, j \in J$ to be open sets and we have to prove that $\mathfrak{i}(\bigcup_{j\in J} A_j) = (\bigcup_{j\in J} A_j)$. But since $\mathfrak{i}(\bigcup_{j\in J} A_j) \subseteq (\bigcup_{j\in J} A_j)$ follows by the second axiom for preinterior, then it is enough to show that $(\bigcup_{j\in J} A_j) \subseteq \mathfrak{i}(\bigcup_{j\in J} A_j)$. For any $j \in J, A_j = \mathfrak{i}(A_j) \subseteq \mathfrak{i}(\bigcup_{j\in J} A_j)$ from Proposition 2. We can take the union over all $j \in J$ in the left-hand side. And therefore $(\bigcup_{j\in J} A_j) \subseteq \mathfrak{i}(\bigcup_{j\in J} A_j)$, which proves the theorem.

For every preinterior operator on $\mathcal{X}$, and any $A \subseteq \mathcal{X}$ we define the *interior* of A,

$$Int_{\mathfrak{i}}(A) = \bigcup\{A_0 \mid A_0 \subseteq A \ \&\ \mathfrak{i}(A_0) = A_0\}. \tag{4}$$

But Theorem 2 implies that $Int_{\mathfrak{i}}(A)$ is open set for every $A \subseteq \mathcal{X}$. And obviously $Int_{\mathfrak{i}}(A)$ is the biggest open set contained in A.

Remark 4 *For every $A \subseteq \mathcal{X}$ and preinterior operator $\mathfrak{i}$ in $\mathcal{X}$,*

$$A \supseteq \mathfrak{i}^0(A) \supseteq \mathfrak{i}^1(A) \supseteq \mathfrak{i}^2(A) \supseteq \cdots \supseteq Int_{\mathfrak{i}}(A).$$

and $Int_{\mathfrak{i}}(A)$ is the biggest open set contained in $\mathfrak{i}^m(A)$ for all $m \in \mathbb{N}$.

Every (pre)closure operator has its correspondant (pre)interior operator and vice versa. Let $\mathfrak{c}$ be a preclosure operator in $\mathcal{X}$ and let us define,

$$\delta(A) := \neg\mathfrak{c}(\neg A).$$

Then δ is a preinterior operator in $\mathcal{X}$, i.e. the axioms for preinterior are satisfied.

1. $\delta(X) = \neg\mathfrak{c}(\neg X) = \neg\mathfrak{c}(\emptyset) = \neg\emptyset = X$;
2. $\mathfrak{c}(\neg A) \supseteq \neg A$ and therefore $A \supseteq \neg\mathfrak{c}(\neg A) = \delta(A)$;
3. $\delta(A \cap B) = \delta(A) \cap \delta(B)$.

Indeed, for the last axiom $\delta(A \cap B) = \neg\mathfrak{c}(\neg(A \cap B)) = \neg\mathfrak{c}(\neg A \cup \neg B)$. But $\mathfrak{c}(\neg A \cup \neg B) = \mathfrak{c}(\neg A) \cup \mathfrak{c}(\neg B)$ and therefore, $\delta(A \cap B) = \neg\mathfrak{c}(\neg A) \cap \neg\mathfrak{c}(\neg B) = \delta(A) \cap \delta(B)$. Moreover, if $\mathfrak{c}$ is a topological closure, i.e. $\mathfrak{c}(A) = \mathfrak{c}^2(A)$ for every $A \in \mathcal{X}$, then δ is also idempotent.

Analogically, if $\mathfrak{i}$ is (pre)interior operator, then $\mathfrak{c}(A) := \neg\mathfrak{i}(\neg A)$ is its corresponding (pre)closure. It is clear now that the family of open sets is composed exactly of the complements of the above defined closed sets if we consider the pair (pre)closure - (pre)interior operators as conjugate pair operators. Obviously, we have shown the validity of the following proposition.

Proposition 3 *If $\mathfrak{i}$ and $\mathfrak{c}$ is a conjugate pair of preinterior and preclosure operators in $\mathcal{X}$, then*

$$\tau^{\mathfrak{c}} = \{\neg A \mid A \in \tau_{\mathfrak{i}}\} \text{ and } \tau_{\mathfrak{i}} = \{\neg B \mid B \in \tau^{\mathfrak{c}}\}.$$

3 Fuzzy (pre)topological operators

In [13] are introduced and discussed the following closure and interior operators for standard fuzzy sets. For every $\alpha \in [0,1]$ we define the closure operator:

$$\mathcal{C}_\alpha : FS(X) \longrightarrow FS(X),$$

such that

$$\mu_{\mathcal{C}_\alpha(A)}(x) = \begin{cases} \mu_A(x) & \text{if } \mu_A(x) \leq \alpha \\ 1 & \text{if } \alpha < \mu_A(x) \end{cases} \tag{5}$$

Actually, in [13] there is an error in the limit of the first equation of the above definition. There is stated that,

$$\mu_{\mathcal{C}_\alpha(A)}(x) = \begin{cases} \mu_A(x) & \text{if } \mu_A(x) \leq \alpha \\ 1 & \text{if } \alpha \leq \mu_A(x), \end{cases}$$

which is not correct since for $0 < \mu_A(x) = \alpha < 1$ we would get that $\mu_{\mathcal{C}_\alpha(A)}(x) = \alpha$ from the first condition and $\mu_{\mathcal{C}_\alpha(A)}(x) = 1$ from the second one. Moreover, the equality sign should really be in the first line in (5). Otherwise, if it was in the second line, if $\alpha = 0$ and $\mu_A \equiv 0$ ($A = \emptyset$) we would get that $\mathcal{C}_0(\emptyset) = X \neq \emptyset$.

For every $\alpha \in [0,1]$ we define the interior operator:

$$\mathcal{I}_\alpha : FS(X) \longrightarrow FS(X),$$

such that

$$\mu_{\mathcal{I}_\alpha(A)}(x) = \begin{cases} 0 & \text{if } \mu_A(x) < \alpha \\ \mu_A(x) & \text{if } \alpha \leq \mu_A(x) \end{cases} \tag{6}$$

In [13] there is again an error in the limit of the first equation of the above definition because the equality sign should be in the second line instead in the first one. If so, in case $\alpha = 1$ we get $\mathcal{I}_1(X) \neq X$, which would violate the first axiom for preinterior operator.

It is clear now that $\mathcal{C}_\alpha(A) = \neg\mathcal{I}_{1-\alpha}(\neg A)$. Therefore, the pair $(\mathcal{C}_\alpha, \mathcal{I}_{1-\alpha})$ is conjugate pair of closure-interior operators defining the same topological structure in $FS(X)$.

Let us generalize the above notions to define preclosure and preinterior operators in $FS(X)$.

For every $\alpha, \gamma \in [0,1]$ we define the preinterior operator:

$$\mathcal{I}_\alpha^\gamma : FS(X) \longrightarrow FS(X),$$

such that

$$\mu_{\mathcal{I}_\alpha^\gamma(A)}(x) = \begin{cases} 0 & \text{if } 0 \le \mu_A(x) < \gamma.\alpha, \\ \frac{1}{1-\gamma}(\mu_A(x) - \alpha) + \alpha & \text{if } \gamma.\alpha \le \mu_A(x) < \alpha, \\ \mu_A(x) & \text{if } \alpha \le \mu_A(x) \le 1. \end{cases} \quad (7)$$

The above definition is well defined even for $\gamma = 1$ (although $\frac{1}{1-\gamma}(\mu_A(x) - \alpha)+\alpha$ is not defined) since in this case the condition in the second expression will become $\alpha < \mu_A(x) \le \alpha$, which is not valid. And therefore it will not be satisfied for any value of $\mu_A(x), x \in X$. Moreover, $\mu_{\mathcal{I}_\alpha^\gamma(A)}(x) = \mu_{\mathcal{I}_\alpha(A)}(x)$ for all $x \in X$, which means that

$$\mathcal{I}_\alpha^1(A) = \mathcal{I}_\alpha(A)$$

Let us check that the above definition is correct. i.e. that $\mathcal{I}_\alpha^\gamma$ satisfies the axioms for pretopology in IFS(X). Indeed, let

$$f(t) = \begin{cases} 0 & \text{if } t < \gamma.\alpha \\ \frac{1}{1-\gamma}(t - \alpha) + \alpha & \text{if } \gamma.\alpha \le t < \alpha \\ t & \text{if } \alpha \le t \end{cases} \quad (8)$$

and it is easily verifiable that $f(t) \le t$ for all $t \in [0, 1]$. This is shown on the first graphic of Figure 1, which represents the function f. Therefore, the second axiom is satisfied. Furthermore, f is continuous and non-decreasing in $[0, 1]$ and therefore we have that $f(\min(t_1, t_2)) = \min(f(t_1), f(t_2))$. This shows that the third axiom is satisfied too ($\mathcal{I}_\alpha^\gamma(A \cap B) = \mathcal{I}_\alpha^\gamma(A) \cap \mathcal{I}_\alpha^\gamma(B)$) and therefore $\mathcal{I}_\alpha^\gamma$ is a preinterior operator in FS(X).

Remark 5 *If* $\gamma = 0$ *then* $f(t) = t$ *and if* $\gamma = 1$ *then*

$$f_{|\gamma=1}(t) = \begin{cases} 0 & \text{if } t < \alpha, \\ t & \text{if } \alpha \le t. \end{cases}$$

and therefore $\mathcal{I}_\alpha^0 = id$ *and* $\mathcal{I}_\alpha^1 = \mathcal{I}_\alpha$ *(see (6)) for all* $\alpha \in [0, 1]$.

The conjugate preclosure operator corresponding to $\mathcal{I}_\alpha^\gamma$ is defined as $\mathcal{C}(A) = \neg\mathcal{I}_\alpha^\gamma(\neg A)$. Let us introduce the function,

$$F(t) = 1 - f(1 - t), \text{ for } t \in [0, 1]$$

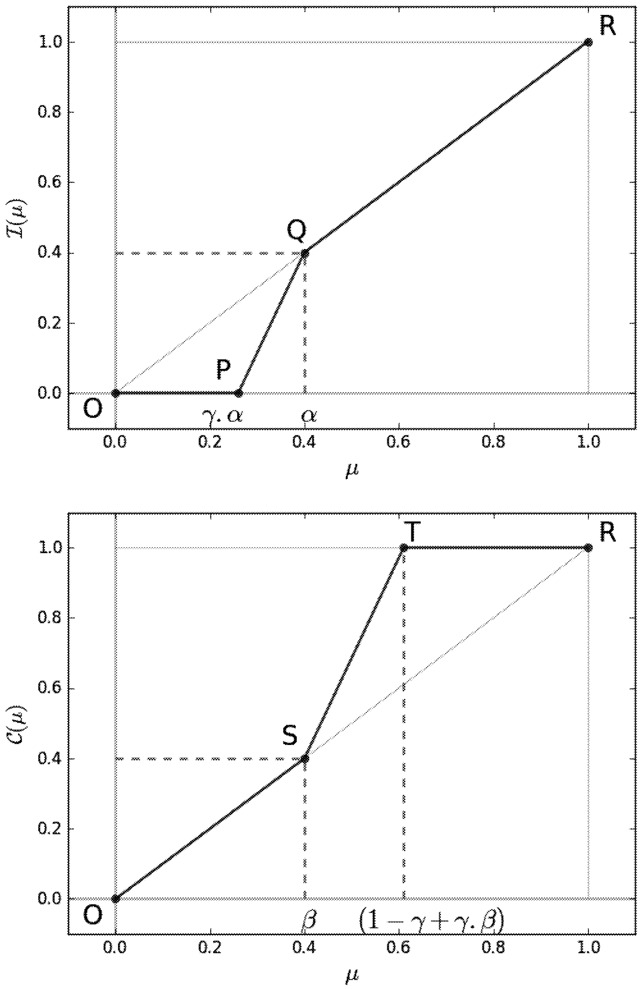

Figure 1: The first graphic represents the the partially continuous map corresponding to the definition of the interior operator $\mathcal{I}_\alpha^\gamma$ from (7). The second graphic represents the map corresponding to the definition of the closure operator $\mathcal{C}_\alpha^\gamma$ from (13).

It is clear now that $\mu_{\mathcal{C}(A)} = F(\mu_A(x))$ if we take $f = \mu_{\mathcal{I}_\alpha^\gamma}$. We have that for $t(x) = 1 - x$

$$f(t) = f(1-x) = \begin{cases} 0 & \text{if } 0 \leq 1-x < \gamma.\alpha, \\ \frac{1}{1-\gamma}(1-x-\alpha)+\alpha & \text{if } \gamma.\alpha \leq 1-x < \alpha, \\ 1-x & \text{if } \alpha \leq 1-x. \end{cases} \tag{9}$$

and hence,

$$f(t) = f(1-x) = \begin{cases} 0 & \text{if } 1-\gamma.\alpha < x \leq 1 \\ \frac{1}{1-\gamma}(1-x-\alpha)+\alpha & \text{if } 1-\alpha < x \leq 1-\gamma.\alpha \\ 1-x & \text{if } x \leq 1-\alpha \end{cases} \tag{10}$$

Finally, for $F(t)$ we have,

$$F(t) = 1 - f(1-t) = \begin{cases} 1 & \text{if } 1-\gamma.\alpha < t \leq 1 \\ \frac{1}{1-\gamma}(t-(1-\alpha))+1-\alpha & \text{if } 1-\alpha < t \leq 1-\gamma.\alpha \\ t & \text{if } t \leq 1-\alpha \end{cases} \tag{11}$$

and hence,

$$\mu_{\mathcal{C}(A)}(x) = \begin{cases} 1 & \text{if } 1-\gamma.\alpha < \mu_A(x) \leq 1 \\ \frac{1}{1-\gamma}(\mu_A(x)-(1-\alpha))+1-\alpha & \text{if } 1-\alpha < \mu_A(x) \leq 1-\gamma.\alpha \\ \mu_A(x) & \text{if } \mu_A(x) \leq 1-\alpha \end{cases} \tag{12}$$

Therefore, if $\beta = 1-\alpha$ we define the preclosure operator $\mathcal{C}_\alpha^\gamma$ as follows,

$$\mu_{\mathcal{C}_\beta^\gamma(A)}(x) = \begin{cases} 1 & \text{if } 1-\gamma+\gamma.\beta < \mu_A(x) \leq 1 \\ \frac{1}{1-\gamma}(\mu_A(x)-\beta)+\beta & \text{if } \beta < \mu_A(x) \leq 1-\gamma+\gamma.\beta \\ \mu_A(x) & \text{if } \mu_A(x) \leq \beta \end{cases}$$

which can be rewritten as (see Figure 1)

$$\mu_{\mathcal{C}_\beta^\gamma(A)}(x) = \begin{cases} \mu_A(x) & \text{if } \mu_A(x) \leq \beta \\ \frac{1}{1-\gamma}(\mu_A(x)-\beta)+\beta & \text{if } \beta < \mu_A(x) \leq 1-\gamma(1-\beta) \\ 1 & \text{if } 1-\gamma(1-\beta) < \mu_A(x) \leq 1 \end{cases} \tag{13}$$

we have that $\mu_{\mathcal{C}(A)} = \mu_{\mathcal{C}_{1-\alpha}^\gamma(A)}$ and the following theorem is satisfied.

Theorem 3 $(\mathcal{C}_{1-\alpha}^{\gamma}, \mathcal{I}_{\alpha}^{\gamma})$ *is a pair of conjugate preclosure-preinterior operators, which define the same topology*

$$\tau_{\mathcal{I}_{\alpha}^{\gamma}} = \{\neg B \mid B \in \tau^{\mathcal{C}_{1-\alpha}^{\gamma}}\} \tag{14}$$

in $FS(X)$.
Moreover, from the definition of $\tau_{\mathcal{I}_{\alpha}^{\gamma}}$ if follows that

$$\tau_{\mathcal{I}_{\alpha}^{\gamma}} = \{A \in FS(X) \mid Range(\mu_A) \subseteq \{0\} \cup [\alpha, 1]\} \tag{15}$$

and

$$\tau^{\mathcal{C}_{\beta}^{\gamma}} = \{B \in FS(X) \mid Range(\mu_B) \subseteq \{1\} \cup [0, \beta]\}. \tag{16}$$

Let us give the following interesting property showing that the topological closure (interior) operators introduced above are limits of their (pre)closure and (pre)interior operators, respectively.

Proposition 4 *For every $\gamma \in (0,1]$ we have that $\lim_{n\to\infty} (\mathcal{C}_{\alpha}^{\gamma})^n = \mathcal{C}_{\alpha}^1 = \mathcal{C}_{\alpha}$. That is the preclosure operator converges to the closure operator $\mathcal{C}_{\alpha}$. Furthermore, from the definition (3) of $Cl_{\mathfrak{c}}(A)$ and the chain of inclusions in Remark 2 we have that for every $A \in FS(X)$,*

$$Cl_{\mathcal{C}_{\alpha}^{\gamma}}(A) = Cl_{\mathcal{C}_{\alpha}}(A) \text{ and } \tau^{\mathcal{C}_{\alpha}^{\gamma}} = \tau^{\mathcal{C}_{\alpha}}.$$

For every $\gamma \in (0,1]$ we have that $\lim_{n\to\infty} (\mathcal{I}_{\alpha}^{\gamma})^n = \mathcal{I}_{\alpha}^1 = \mathcal{I}_{\alpha}$. That is the preinterior operator converges to the interior operator $\mathcal{I}_{\alpha}$. Furthermore, from the definition (4) of $Int_{\mathfrak{i}}(A)$ and the chain of inclusions in Remark 4 we have that for every $A \in FS(X)$,

$$Int_{\mathcal{I}_{\alpha}^{\gamma}}(A) = Int_{\mathcal{I}_{\alpha}}(A) \text{ and } \tau_{\mathcal{I}_{\alpha}^{\gamma}} = \tau_{\mathcal{I}_{\alpha}}.$$

Proof We have to show that for every $A \in FS(X)$ and $x \in X$,

$$\lim_{n\to\infty} \mu_{(\mathcal{I}_{\alpha}^{\gamma})^n(A)}(x) = \mu_{\mathcal{I}_{\alpha}^1(A)}(x).$$

The above expressions follow geometrically from Figure 1. But let us write them out analytically too.
It is enough to check it for $\mu_A(x) < \alpha$ since for $\mu_A(x) \geq \alpha$ it is trivial by the definition of $\mathcal{I}_{\alpha}^{\gamma}$. If $\mu_A(x) \leq \gamma.\alpha$, then $\mu_{\mathcal{I}_{\alpha}^{\gamma}(A)}(x) = 0$ and since $\mu_A(x) \leq \gamma.\alpha \leq \alpha$, then $\mu_{\mathcal{I}_{\alpha}^1(A)}(x) = 0$ and $\mu_{\mathcal{I}_{\alpha}^{\gamma}(A)}(x) = \mu_{\mathcal{I}_{\alpha}^1(A)}(x)$. Suppose now that $\gamma.\alpha < \mu_A(x) < \alpha$. Then since

$$\mu_{(\mathcal{I}_{\alpha}^{\gamma})^n(A)} = \frac{1}{(1-\gamma_{\alpha})^n}(\mu_A(x) - \alpha) + \alpha,$$

we are looking for $n \in \mathbb{N}$ such that the above expression is less or equal than $\gamma.\alpha$. This would mean, that, $\mu_{(\mathcal{I}_\alpha^\gamma)^{n+1}(A)}(x) = 0 = \mu_{(\mathcal{I}_\alpha^1)(A)}(x)$. We have to show that $\frac{1}{(1-\gamma)^n}(\mu_A(x) - \alpha) + \alpha \leq \gamma.\alpha$ or $\alpha(1-\gamma) \leq \frac{1}{(1-\gamma)^n}(\alpha - \mu_A(x))$, which equals $(1-\gamma)^{n+1} \leq \frac{\alpha - \mu_A(x)}{\alpha}$. But since $0 < \frac{\alpha-\mu_A(x)}{\alpha} < 1$ and $0 < 1-\gamma < 1$, taking the natural logarithm on both sides of the above inequality we obtain $(n+1)\ln(1-\gamma) \leq \ln\frac{\alpha-\mu_A(x)}{\alpha}$. And since both sides are negative numbers it is equivalent to,

$$n + 1 \geq \frac{\ln\frac{\alpha-\mu_A(x)}{\alpha}}{\ln(1-\gamma)}.$$

Therefore, for $\gamma.\alpha < \mu_A(x) < \alpha$ and $n \geq \frac{\ln\frac{\alpha-\mu_A(x)}{\alpha}}{\ln(1-\gamma)} - 1$ we obtain again the target equality $\mu_{(\mathcal{I}_\alpha^\gamma)^n(A)}(x) = \mu_{\mathcal{I}_\alpha^{1,1}(A)}(x)$. This leads to the proof of our initial assumption that for every $\gamma \in (0,1]$ we have that $\lim\limits_{n\to\infty}(\mathcal{I}_\alpha^\gamma)^n = \mathcal{I}_\alpha^1$. The above statement and Remark 4 provide that $Int_{\mathcal{I}_\alpha^\gamma}(A) = Int_{\mathcal{I}_\alpha}(A)$ for any $A \in FS(X)$ and therefore $\tau_{\mathcal{I}_\alpha^\gamma} = \tau_{\mathcal{I}_\alpha}$.
The statements for the counterparts - the preclosure and closure operators follow by analogy.

4 Conclusion

In this paper we have introduced a concrete example of fuzzy pretopology and fuzzy topology through the (pre)interior ($\mathcal{I}_\alpha^\gamma$) and (pre)closure ($\mathcal{C}_\alpha^\gamma$) operators. The mathematical correctness of the operators has also been proved and we showed in Theorem 3 that these operators are conjugate. The topological operators have been expressed as limits of their pretopological counterparts. Tese (pre)topological operators can be adequately applied in real world problems like in GIS, as showed in [13]. The results from the current paper will be further extended in two directions - application in machine learning like clustering and classification and, generalization in the framework of intuitionistic fuzzy sets.

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